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Topological Massive Sigma Models

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ABSTRACT

In this paper we construct topological sigma models which include a potential and are related to twisted massive supersymmetric sigma models. Contrary to a previous construction these models have no central charge and do not require the manifold to admit a Killing vector. We use the topological massive sigma model constructed here to simplify the calculation of the observables. Lastly it is noted that this model can be viewed as interpolating between topological massless sigma models and topological Landau-Ginzburg models.

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1. Introduction

Ever since Witten's pioneering work [1,2] there has been a great deal of interest in topological field theories, both as a tool for calculating topological invariants of manifolds and as possible physical theories describing a "phase" of quantum gravity where there are no local gravitational degrees of freedom. In particular, the topological sigma model [2] is hoped to describe an unbroken "phase" of string theory occurring in a high energy limit. However, if topological theories are to be of direct physical relevance, their topological invariance must be spontaneously broken by some, as of yet unknown, mechanism. In order to look for symmetry breaking mechanisms, it is of interest to construct topological theories which contain nontrivial potentials. In addition, the calculation of some observables in the topological sigma model may be simplified by using a massive model in the (exact) semi-classical limit that the mass tends to infinity. Finally, the relationship between massive sigma models and Landau-Ginzburg theories has received substantial interest recently; we will see here that the similarities between these two theories are perhaps most transparent in their topological phases.

In [3] an attempt was made to include potential terms for the topological sigma model by twisting the massive (2,2) supersymmetric sigma model. However, the potential employed there was expressed as the length of a Killing vector on the target manifold. This form of potential cannot be defined for arbitrary choices of the metric on a manifold with a given topology and as such is somewhat unsatisfactory. We begin here by constructing a new topological sigma model with potential by considering the twisted massive (2,0) supersymmetric sigma model, for which no Killing condition on the potential is required. We briefly discuss the discrepancies between the results presented here and those obtained in [3] and the role of central charges in the twisted algebra. Finally we calculate the vacuum expectation value of some observables and note that the topological massive sigma model constructed here may be interpreted as interpolating between the topological massless sigma model of [2] and the topological Landau-Ginzburg model of [5].

2. Constructing Topological Field Theories

In general, to construct a topological field theory one needs a scalar Grassmann operator Q such that $Q^2 = 0$, which we interpret as a BRST operator representing a symmetry $\mathcal{S}$. We require that all physical states $\mid phys >$ are invariant under $\mathcal{S}$, hence $Q \mid phys >= 0$. Moreover, two physical states are identified if they differ by a Q-exact state. Thus the physical states correspond to Q-cohomology classes. The observables $\mathcal{O}$ are required to satisfy $\{Q, \mathcal{O}\} = 0$ and $\delta\mathcal{O}/\delta g^{\mu\nu} = \{Q, \mathcal{K}_{\mu\nu}\}$ for some $\mathcal{K}_{\mu\nu}$, where $g_{\mu\nu}$ is the spacetime metric and $\{\ ,\ \}$ is a $\mathbf{Z}_2$ graded commutator [6]. Two observables are identified if their difference is Q-exact and therefore they also correspond to Q-cohomology classes.

If we take the action of the theory to be a Q-commutator

$$S = \{Q, V\} \tag{2.1}$$

for some V, then it follows that $\{Q, S\} = \{Q^2, V\} = 0$ and the action is invariant under $\mathcal{S}$. Furthermore, if Q does not depend on the metric $g_{\mu\nu}$ then the energy-momentum tensor of the theory is Q-exact

$$T_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}} = \{Q, \frac{\delta V}{\delta g^{\mu\nu}}\}\ . \tag{2.2}$$

The expectation value of an observable is defined via the Feynman Path integral as

$$< \mathcal{O} > \equiv \int \mathcal{O} e^{-S}\ . \tag{2.3}$$

In (2.3) we must specify what the boundary conditions are for the various fields. While the boundary conditions for bosonic fields are always even, those for the fermionic fields can be either even or odd. Often however, the Grassmann fields of a topological theory are interpreted as ghosts and so do not obey fermionic statistics, as will be the case here. We will therefore employ periodic boundary conditions for all the fields in (2.3).

It follows from the above definitions that the expectation value of an observable is metric independent since

$$
\begin{aligned}
\frac{\delta}{\delta g^{\mu\nu}} < \mathcal{O} > &= \frac{\delta}{\delta g^{\mu\nu}} \int \mathcal{O} e^{-S} \\
&= \int \left(\frac{\delta \mathcal{O}}{\delta g^{\mu\nu}} - \mathcal{O} T_{\mu\nu} \right) e^{-S} \\
&= < \left\{ Q, \mathcal{K}_{\mu\nu} - \mathcal{O} \frac{\delta V}{\delta g^{\mu\nu}} \right\} > \\
&= 0 \ .
\end{aligned}
\tag{2.4}
$$

Therefore there are no dynamics in the system. In addition, if Q does not depend on a particular coupling g, then the expectation values of any observable also do not depend on g. This can easily be seen by an identical argument to that used in (2.4). Assuming that the path integral measure is topologically invariant (i.e. there are no anomalies) then if the above conditions are satisfied we have a topological field theory.

In the topological sigma model the target space metric g_{ij} plays the role of a coupling on which Q explicitly depends. Thus the above argument does not suffice to prove invariance with respect to g_{ij}. However, if under a metric variation V changes by $\delta V = V^{ij} \delta g_{ij}$ and $Q \mid phys >= 0$ for any choice of g_{ij} used in constructing Q, then δS is always a Q commutator as

$$
\begin{aligned}
\delta S &= \{Q, \delta V\} + \{\delta Q, V\} \\
&= \{Q, \delta V - V\} + \{Q', V\} \ ,
\end{aligned}
\tag{2.5}
$$

where $\delta Q = Q' - Q$ and $Q \mid phys >= Q' \mid phys >= 0$. Hence, provided δV can be defined, the variation of any target space metric invariant observable has zero expectation value.

In [3] a twisted version of the massive topological sigma model was constructed which required the manifold to admit a Killing vector. Hence the number of manifolds for which a potential can be defined is limited and the metric deformations

must be suitably restricted. Furthermore the Q generator used in [3] was not nilpotent. Instead Q^2 acts as the Lie derivative along the Killing vector. While it is possible to define Q cohomology when $Q^2 \neq 0$ using the methods of [7], we will avoid this issue here by maintaining $Q^2 = 0$.

3. Twisting the massive (2,0) model

The (2,0) supersymmetry algebra consists of two hermitian supersymmetry generators Q_+^a, $a = 1, 2$ which we combine into a single complex generator $Q_+ = \frac{1}{\sqrt{2}}(Q_+^1 + iQ_+^2)$ and its hermitian conjugate $\bar{Q}_+ = \frac{1}{\sqrt{2}}(Q_+^1 - iQ_+^2)$. We note that in two dimensions it is possible to have left and right handed vectors, which we denote by $\neq$ and $=$ subscripts respectively, as these are just "self-dual" and "anti-self-dual" conditions. The subscripts, counted as 1 and -1 respectively, also indicate the "Lorentz charge"*. Together with the momentum generators $P_{\neq}$ and $P_{=}$ we have the (2,0) supersymmetry algebra

$$\begin{aligned}
&\{Q_+, \bar{Q}_+\} = P_{\neq} \ , \qquad && Q_+^2 = \bar{Q}_+^2 = 0 \ , \\
&[J, Q_+] = \frac{1}{2} Q_+ \ , && [J, \bar{Q}_+] = \frac{1}{2}\bar{Q}_+ \ , \\
&[J, P_{=}] = -P_{=} \ , && [J, P_{\neq}] = P_{\neq} \ , \\
&[U, Q_+] = Q_+ \ , && [U, \bar{Q}_+] = -\bar{Q}_+ \ , \\
&[U, J] \ = \ [U, P_{=}] \ = \ [U, P_{\neq}] \ = \ 0 \ .
\end{aligned} \tag{3.1}$$

Here, J is the generator of two dimensional Lorentz transformations and U generates an internal $SO(2)$ rotation between Q_+^1 and Q_+^2. While only left handed generators appear in (3.1) U can be split into left and right components $U = U_L \otimes U_R$ corresponding to the left and right handed modes on the world sheet. Under the generators $J \otimes U_L \otimes U_R$, $(Q_+, \bar{Q}_+)$ transform as $(\frac{1}{2}, 1, 0) \oplus (\frac{1}{2}, -1, 0)$. If we twist

$\star$ On a Riemann surface these become holomorphic and antiholomorphic vectors respectively.

(3.1) by identifying a new generator of Lorentz transformations

$$T = J - \frac{1}{2}U_L + \frac{1}{2}U_R \ , \tag{3.2}$$

then, with respect to $T \otimes U_L \otimes U_R$, $(Q_+, \bar{Q}_+)$ transform as $(0,1,0) \oplus (1,-1,0)$. Thus we have a scalar generator, which we denote by Q and a left handed vector generator, which we denote by $\bar{Q}_{\neq}$. The new algebra takes the form

$$\begin{aligned}
&\{Q, \bar{Q}_{\neq}\} = P_{\neq} \ , \qquad && Q^2 = \bar{Q}_{\neq}^2 = 0 \ , \\
&[T, Q] = 0 \ , && [T, \bar{Q}_{\neq}] = \bar{Q}_{\neq} \ , \\
&[T, P_{=}] = -P_{=} \ , && [T, P_{\neq}] = P_{\neq} \ , \\
&[U, Q] = Q \ , && [U, \bar{Q}_{\neq}] = -\bar{Q}_{\neq} \ , \\
&[U, T] \ = \ [U, P_{=}] \ = \ [U, P_{\neq}] \ = \ 0 \ .
\end{aligned} \tag{3.3}$$

We have therefore identified a generator Q which we may use as the BRST operator for topological symmetry. This is the so called "A" twist. Another possibility is the "B" twist $T = J - \frac{1}{2}U_L - \frac{1}{2}U_R$ in which case the algebra (3.3) remains the same but the twisted fields change accordingly. Before we address the problem of finding a set of fields for the algebra (3.3) to act on, we give a short description of the massive (2,0) supersymmetric sigma model.

The massive (2,0) supersymmetric sigma model is defined by scalar maps ϕ^i, and their anticommuting spinor superpartners λ_+^i, which map from a two dimensional Minkowski spacetime Σ (the base space) into an arbitrary complex manifold $\mathcal{M}$ of (real) dimension D (the target space). $\mathcal{M}$ is endowed with an hermitian metric g_{ij}, complex structure $J^i{}_j$ and antisymmetric tensor b_{ij}. In addition, there is an anticommuting spinor field ζ_-^a which maps from Σ into an arbitrary complex vector bundle Ξ over $\mathcal{M}$ with hermitian metric h_{ab} and connection $A_i^a{}_b$. To include a potential for the fields we introduce a section s^a of the bundle Ξ, which is the sum of a holomorphic and an antiholomorphic section. The action given in

complex coordinates is

$$S_{(2,0)} = \int d^2x \left\{ (g_{I\bar{J}} + b_{I\bar{J}})\partial_{\neq}\phi^I \partial_{=}\phi^{\bar{J}} + ig_{I\bar{J}}\lambda_+^I \nabla_{=}^{(+)}\lambda_+^{\bar{J}} - ih_{A\bar{B}}\zeta_-^A \hat{\nabla}_{\neq}\zeta_-^{\bar{B}} \right.$$
$$\left. -\frac{1}{2}\zeta_-^A\zeta_-^{\bar{B}} F_{I\bar{J}}^{A\bar{B}}\lambda_+^I\lambda_+^{\bar{J}} + mh_{A\bar{B}}\hat{\nabla}_I s^A \lambda_+^I \psi_-^{\bar{B}} - \frac{1}{4}m^2 h_{A\bar{B}} s^A s^{\bar{B}} \right\} . \tag{3.4}$$

Here $F_{I\bar{J}}^{A\bar{B}}$ is the curvature of the connection $A_{i\ b}^{a}$, $\hat{\nabla}$ is the covariant derivative with respect to $A_{i\ b}^{a}$ while $\nabla^{(+)}$ is the covariant derivative with respect to the Levi-Civita connection with torsion $H_{ijk} = \frac{3}{2}\partial_{[i}b_{jk]}$.

In this paper we will be concerned primarily with the twisted version of the (2,0) supersymmetric model (3.4). We may consider the (2,2) supersymmetric model, as was done in [3], by taking the special case [8] where we identify Ξ with $T\mathcal{M}$ and $A_{i\ b}^{a}$ with the spin connection ω_i^{jk} by introducing a vielbein ${e_i}^a$. Furthermore, in the (2,2) supersymmetric case, the section must be defined by a holomorphic Killing vector X^i to be

$$s^a = {e_i}^a(u^i - X^i) \ , \tag{3.5}$$

where $\partial_{[i}u_{j]} = X^k H_{ijk}$. The presence of the left handed supersymmetries, however, induces central charges in the algebra (3.1) given by the derivative of the Killing vector. Thus, with regards to the discussion above, the (2,2) model with potential (3.5) cannot be twisted into a topological theory as is done in [3], unless $X^i = 0$. In this case (locally) $u_i = \partial_i f$ for some scalar f. Contrary to the claim in [3,4], it is possible to twist this theory and the result is a special case of the model constructed below, provided we interpret the scalar f as a worldsheet 1-form, i.e. $f \in \Phi_*(\Lambda^1(\Sigma))$. In this way Lorentz invariance is maintained in the twisted model. In order to construct a topological version of the massive sigma model (3.4) we will find it necessary make the identification $\Xi \equiv T\mathcal{M}$, $A_{i\ b}^{a} = {e_j}^a e_{k\ b}\omega_i^{jk}$. For simplicity we will also assume that $\mathcal{M}$ is compact in what follows. However, to prevent the appearance of central charges and Killing vectors, we do not enforce (3.5). It is therefore possible to view the topological model below as a twisted

version of the massless (2,2) supersymmetric sigma model, with a potential which breaks the supersymmetry down to (2,0).

Our next step in the construction is to replace the fields $(\phi^I, \phi^{\bar{I}}, \lambda^I_+, \lambda^{\bar{I}}_+, \zeta^I_-, \zeta^{\bar{I}}_-)$ by their twisted counterparts which we denote as $(\phi^I, \phi^{\bar{I}}, \eta^I, \psi^{\bar{I}}_{\neq}, \psi^I_{=}, \eta^{\bar{I}})$. Under the generators $T \otimes U$ these fields transform as $(0,0) \oplus (0,0) \oplus (0,1) \oplus (\frac{1}{2}, -1) \oplus (-\frac{1}{2}, -1) \oplus (0,1)$. λ^I_+ and $\psi^{\bar{I}}_-$ are now worldsheet scalars η^I and $\eta^{\bar{I}}$, while $\eta^{\bar{I}}_+$ and ζ^I_- are twisted into the $\psi^{\bar{I}}_{\neq}$ and $\psi^I_{=}$ components of a worldsheet 1-form, $\psi^i \in \Lambda^1(\Sigma) \otimes \Phi^*(T\mathcal{M})$. The subscripts $=$ and $\neq$ can be viewed as referring to the (1,0) and (0,1) components of 1-forms on Σ respectively. Here we have performed an "A" twisting described above. Had we used the "B" twist we would arrive at an action similar to that in [3]. The proof that the twisted version of a theory is indeed topological requires us to explicitly write the action in the form of equation (2.1). It is therefore necessary to find the action of Q on the twisted fields.

At this point we may generalize the construction to non Hermitian manifolds. To this end we simply postulate a set of fields (ϕ^i, η^i, ψ^i), where ϕ^i, η^i are scalars and ψ^i a 1-form with components $\psi^i_{=}$ and $\psi^i_{\neq}$, along with their transformations under Q. In order to close the Q-algebra off-shell it is necessary to introduce a commuting, non propagating 1-form field $H^i \in \Lambda^1(\Sigma) \otimes \Phi^*(T\mathcal{M})$ with components $H^i_{=}$ and $H^i_{\neq}$, transforming under the action of $T \otimes U_L \otimes U_R$ as $(-1,0,0)$ and $(1,0,0)$ respectively. Furthermore ψ^i and H^i are self dual in the sense that $\psi^i_{=} = -iJ^i{}_j \psi^j_{=}$, $\psi^i_{\neq} = iJ^i{}_j \psi^j_{\neq}$ and similarly for H^i. In the special case that $\mathcal{M}$ is an Hermitian complex manifold these constraints are solved by setting $\psi^I_{\neq} = \psi^{\bar{I}}_{=} = 0$ whereby we recover the above twisted fields.

The action of the generator Q is the same as the standard topological sigma model [2]:

$$
\begin{aligned}
[Q,\phi^i] &= i\eta^i \ , \\
\{Q,\eta^i\} &= 0 \ , \\
\{Q,\psi^i\} &= H^i + \frac{1}{2} i\nabla_j J^i{}_k \eta^j \psi^k - i\Gamma^i_{jk}\eta^j\psi^k \ , \\
[Q,H^i] &= \frac{1}{2} i\nabla_k J^i{}_j \eta^k H^j - i\Gamma^i_{jk}\eta^j H^k - \frac{1}{4}(\nabla_j J^i{}_m)(\nabla_k J^m{}_l)\eta^j\eta^k\psi^l \\
&\quad - \frac{1}{2}(R^i{}_{jkl} - \nabla_k\nabla_l J^i{}_j)\eta^k\eta^l\psi^j \ .
\end{aligned}
\tag{3.6}
$$

The first commutator represents the symmetry of an arbitrary shift in the coordinate ϕ^i of $\mathcal{M}$, while the second is necessary for $Q^2 = 0$. The third anticommutator may be taken as the definition of the non propagating field H^i while the condition $\{Q^2, \psi^i\} = 0$ determines the commutator $[Q, H^i]$ uniquely. That $[Q^2, H^i]$ vanishes follows from a lengthy but straightforward calculation.

In the twisted algebra (3.3) we interpret the U invariance of the theory as a ghost symmetry and call the corresponding quantum number of a field its ghost number. Therefore, as $[Q, U] = Q$, the action of U on a field raises its ghost number by one. We wish to construct Lorentz invariant theories which preserve the U symmetry, so that both the T and J symmetries are preserved. Hence we must find a topologically invariant Lagrangian with ghost number 0. In addition, if the theory is to be topological with respect to the worldsheet Σ, the Lagrangian must certainly be conformally invariant. The scalars ϕ^i and η^i have conformal dimension 0 while the worldsheet 1-forms ψ^i and H^i have conformal dimension 1. The properties of the various fields are summarized in table 1 below.

Field	Statistics	Conformal Dimension	Ghost Number
ϕ^i	$+$	0	0
η^i	$-$	0	1
ψ^i	$-$	1	-1
H^i	$+$	1	0

Table 1: Properties of the Twisted Fields.

In the (2,0) supersymmetric sigma model (3.4) the mass term breaks conformal invariance at the classical level and so one may suspect that we can not construct a topological version of this theory. However, in order to maintain Lorentz invariance in the twisted model, we must interpret s^i as a tangent space valued, worldsheet 1-form; $s^i \in \Lambda^1(\Sigma) \otimes \Phi^*(T\mathcal{M})$, with components $s^i_=$ and $s^i_{\neq}$. In this case s^i has conformal dimension 1 and the "mass" parameter m is dimensionless.

In order to construct a topological version of the theory (3.4) all we have to do now is specify a suitable scalar function V in (2.1) which has ghost number -1 and conformal dimension 0. A sufficiently general choice is

$$V = \int d^2x \left\{ g_{ij}\psi^i_= \partial_{\neq}\phi^j - \alpha g_{ij}\psi^i_= H^j_{\neq} - m g_{ij}\psi^i_= s^j_{\neq} \ + (=\leftrightarrow\neq) \right\} \ , \tag{3.7}$$

where α and m are dimensionless constants. An alternative, but equivalent construction could have been made by redefining $H^i \to H^i + \frac{m}{\alpha}s^i$ in (3.3) and dropping the last term in (3.7). We must then modify the Q commutators correspondingly, but this is easily done and automatically maintains $Q^2 = 0$.

The topological action derived from (3.7) using (2.1) and (3.6) is

$$\begin{aligned} S_{top} = \int d^2x \Big\{ & -2\alpha g_{ij} H^i_= H^j_{\neq} + g_{ij}(\partial_= \phi^i - m s^i_=) H^j_{\neq} + g_{ij}(\partial_{\neq}\phi^i - m s^i_{\neq}) H^j_= \\ & -\alpha \psi^i_= \psi^j_{\neq} (R_{ijkl} - \nabla_k \nabla_l J_{ij} + \frac{1}{2}\nabla_k J_{im} \nabla_l J^m_{\ j}) \eta^k \eta^l \\ & -i g_{ij}\psi^i_= (\nabla_{\neq}\eta^j + \frac{1}{2}\nabla_k J^j_{\ l} \partial_{\neq}\phi^l \eta^k) - i g_{ij}\psi^i_{\neq}(\nabla_= \eta^j + \frac{1}{2}\nabla_k J^j_{\ l}\partial_= \phi^l \eta^k) \\ & -i m g_{ij}(\nabla_k s^i_{\neq} + \frac{1}{2}\nabla_k J^i_{\ m} s^m_{\neq}) \eta^k \psi^j_= \\ & -i m g_{ij}(\nabla_k s^i_= + \frac{1}{2}\nabla_k J^i_{\ m} s^m_=) \eta^k \psi^j_{\neq} \Big\} \ . \end{aligned} \tag{3.8}$$

Here we see that H^i is indeed non-propagating. If we remove H^i by its equation of motion (keeping its self duality in mind), (3.8) becomes the more familiar sigma

model action

$$
\begin{aligned}
&S_{top} = \\
&\int d^2x \left\{ \frac{1}{2\alpha}(g_{ij} + iJ_{ij})\partial_{\neq}\phi^i \partial_{=}\phi^j - \alpha \psi^i_{=} \psi^j_{\neq} (R_{ijkl} - \nabla_k \nabla_l J_{ij} + \frac{1}{2}\nabla_k J_{im} \nabla_l J^m_{\ j})\eta^k \eta^l \right. \\
&\qquad - i g_{ij} \psi^i_{=} (\nabla_{\neq}\eta^j + \frac{1}{2}\nabla_k J^j_{\ l} \partial_{\neq}\phi^l \eta^k) - i g_{ij} \psi^i_{\neq} (\nabla_{=}\eta^j + \frac{1}{2}\nabla_k J^j_{\ l} \partial_{=}\phi^l \eta^k) \\
&\qquad - i m g_{ij} (\nabla_k s^i_{\neq} + \frac{1}{2}\nabla_k J^i_{\ m} s^m_{\neq}) \eta^k \psi^j_{=} - i m g_{ij} (\nabla_k s^i_{=} + \frac{1}{2}\nabla_k J^i_{\ m} s^m_{=}) \eta^k \psi^j_{\neq} \\
&\qquad \left. - \frac{m}{2\alpha}(g_{ij} + iJ_{ij})(s^i_{=}\partial_{\neq}\phi^j + s^i_{\neq}\partial_{=}\phi^j) + \frac{m^2}{2\alpha}(g_{ij} + iJ_{ij}) s^i_{=} s^j_{\neq} \right\} .
\end{aligned}
\tag{3.9}
$$

So far we have implicitly assumed that the worldsheet metric $g_{\mu\nu}$ is flat. However, as all the formulas we have written are in terms of differential forms and the action is conformally invariant, we can extend the model to be defined on an arbitrary Riemann base manifold Σ [2]. Furthermore, as is the case with the massless topological sigma model [2], it is not necessary for the Nijenhuis tensor to vanish in order to close the topological algebra (3.6). Hence the manifold $\mathcal{M}$ need only be almost complex. In addition, no restrictions are required on the section s^i. We may therefore interpret (3.9) as a more general model defined for an almost complex manifold $\mathcal{M}$. This generalized model cannot arise from twisting the massive sigma model (3.4), since the supersymmetry algebras (3.1) and (3.3) are no longer satisfied.

Consider the case where $\mathcal{M}$ is complex with an Hermitian metric. The action (3.9) then reduces to a twisted version of the model (3.4) (with $b_{ij} = 0$ and Ξ identified with $T^*\mathcal{M}$), with an additional mass term $\frac{m}{2\alpha} g_{I\bar{J}}(s^I_{=}\partial_{\neq}\phi^{\bar{J}} + \partial_{=}\phi^I s^{\bar{J}}_{\neq})$. The mass terms therefore do not entirely arise from simply twisting the massive sigma model (3.4). We will see below however, that the appearance of this additional mass term allows the topological massive sigma model to be identified with the topological Landau-Ginzburg model in the limit $m \to \infty$. The Q commutators (3.6) rely on the identification of Ξ with $T^*\mathcal{M}$ so that the massless version of (3.4) admits (2,2) supersymmetry. We may arrive at a model related to the twisted

(2,0) supersymmetric sigma model by setting $\psi^I_{=} = \eta^{\bar{I}} = 0$. We shall discuss this case in more detail in the next section. In all these cases by setting $m = 0$ we obtain topological twisted versions of the (2,2) and (2,0) supersymmetric sigma sigma models first constructed in [2]. In the special case that $s^i = \partial^i f$ where f is a 1-form on Σ pushed forward to $\mathcal{M}$ by ϕ^i, the action (3.9) is related to a twisted version of the (2,2) supersymmetric sigma model (3.4) with the Killing vector equal to zero.

It is clear from the discussion earlier that the stress-energy tensor of this theory is Q-exact and is given by $T_{\mu\nu} = \{Q, \delta V/\delta g^{\mu\nu}\}$. Hence the theory is topological with respect to the worldsheet metric. For completeness we also show explicitly that $\delta S/\delta g_{ij}$ is Q-exact so that the theory is topological with respect to the target space metric. A straight forward calculation shows that

$$\delta S = \left\{ Q, \int d^2x (\psi^i_{=} \partial_{\neq} \phi^j - m\psi^i_{=} s^j_{\neq} \delta g_{ij} + {=}{\leftrightarrow}{\neq}\ ) \right\} \ . \tag{3.10}$$

4. Observables

In the massless topological sigma model an interesting class of observables can be defined by an n-form $A_{i_1 \ldots i_n} d\phi^{i_1} \ldots d\phi^{i_n}$ on $\mathcal{M}$ [2] by

$$\mathcal{O}^{(0)}_A = A_{i_1 \ldots i_n} \eta^{i_1} \ldots \eta^{i_n} \ . \tag{4.1}$$

Then we have $\{Q, \mathcal{O}^{(0)}_A\} = -i\mathcal{O}^{(0)}_{d_{\mathcal{M}} A}$, where $d_{\mathcal{M}}$ is the exterior derivative on $\mathcal{M}$. Hence $\mathcal{O}^{(0)}_A$ is a BRST observable if and only if $d_{\mathcal{M}} A = 0$. Furthermore, if two n-forms lie in the same cohomology class, they represent the same observable. Therefore the observables $\mathcal{O}^{(0)}_A$ are in a one to one correspondence with the cohomology group $H^n(\mathcal{M})$.

In addition the (closed) n-form A defines two more observables $\mathcal{O}_A^{(1)}$ and $\mathcal{O}_A^{(2)}$ defined over 1-cycles and 2-cycles in Σ respectively. They are defined as

$$
\begin{aligned}
\mathcal{O}_A^{(1)} &= nA_{i_1 \ldots i_n} d\phi^{i_1}\eta^{i_2} \ldots \eta^{i_n} \ , \\
\mathcal{O}_A^{(2)} &= \frac{n(n-1)}{2} A_{i_1 \ldots i_n} d\phi^{i_1} d\phi^{i_2} \eta^{i_3} \ldots \eta^{i_n} \ ,
\end{aligned}
\tag{4.2}
$$

It is not hard to show [2] that $d_\Sigma \mathcal{O}_A^{(k)}$ is Q-exact, where d_Σ is the exterior derivative on Σ, and $\{Q, \mathcal{O}_A^{(k)}\} = -i\mathcal{O}_{d_{\mathcal{M}}A}^{(k)} = 0, \ k = 1, 2$. Therefore, by integrating over a 1-cycle γ and 2-cycle β in Σ, one obtains the observables

$$
\begin{aligned}
\mathcal{W}_1(\gamma) &= \int_\gamma \mathcal{O}_A^{(1)} \ , \\
\mathcal{W}_2(\beta) &= \int_\beta \mathcal{O}_A^{(2)} \ .
\end{aligned}
\tag{4.3}
$$

It also follows that for $k = 1, 2$

$$
\mathcal{W}_k(\partial\gamma) = \int_{\partial\gamma} \mathcal{O}_A^{(k)} = \int_\gamma d_\Sigma \mathcal{O}_A^{(k)}
\tag{4.4}
$$

is Q-exact. Hence the expectations values of $\mathcal{W}_1(\gamma)$ and $\mathcal{W}_2(\beta)$ depend only on the de Rahm cohomology class of A and the homology classes of the cycles γ and β respectively.

It follows from the arguments in section 2 that the expectation values of the observables are independent of the parameters α and m. Therefore we can take the limit $m \to 0$ and recover massless topological sigma model, or alternatively, take $m \to \infty$, where the action simplifies and observables can be more readily calculated.

To this end we will now compute the expectation values of the observables $\mathcal{O}_A^{(0)}$ explicitly using the mass term to simplify the work. First let us consider the "(2,0)

model" in complex coordinates where $g_{I\bar{J}}$ is Hermitian and we set $\psi^I_= = H^I_= = \eta^{\bar{I}} = 0$. For further simplicity we work in the $\alpha = 0$ "gauge":

$$S_{top} = \int d^2x \left\{ g_{I\bar{J}}(\partial_= \phi^I - m s^I_=) H^{\bar{J}}_{\neq} - i g_{\bar{I}J} \psi^{\bar{I}}_{\neq} \nabla_= \eta^J \right. \\ \left. - i m g_{I\bar{J}} \nabla_K s^I_= \eta^K \psi^{\bar{J}}_{\neq} \right\} \ . \tag{4.5}$$

The expectation value of $\mathcal{O}^{(0)}_A$ is defined as

$$< \mathcal{O}^{(0)}_A >= \int d[H] d[\phi] d[\eta] d[\phi] \mathcal{O}^{(0)}_A e^{-S_{top}} \ . \tag{4.6}$$

To evaluate (4.6) we may invoke the use of a Nicoli map [6], by making a change of variables to

$$\pi^I_= = \partial_= \phi^I - m s^I_= \tag{4.7}$$

We must be careful here not to include any zero modes to insure that the transformation is well defined. Therefore we leave out the zero modes ϕ^I_0 and integrate over them separately. We shall elaborate on them shortly. The above transformation has the effect of trivializing the first term in (4.5) to $g_{I\bar{J}} \pi^I_= H^{\bar{J}}_{\neq}$ and introducing a Jacobian factor $\mid \det'(B^I_{=\ J}) \mid$, where $B^I_{=\ J} = \delta \pi^I_= / \delta \phi^J$, into the measure:

$$< \mathcal{O}^{(0)}_A >= \int\limits_M d\phi_0 \int d[H] d[\psi] d[\eta] d[\pi] \frac{\mathcal{O}^{(0)}_A}{\mid \det'(B^I_{=\ J}) \mid} e^{-S_{top}} \ . \tag{4.8}$$

In (4.8) M is the moduli space of bosonic zero modes of $B^I_{=\ J}$ and the prime indicates that we omit the zero modes in calculating the determinant. The Jacobian can be found as the first order term of $\pi^I_=$ in a background field expansion of ϕ^I [6]. This yields

$$\phi^I \to \phi^I + \xi^I \\ \pi^I_= \to \partial_= \phi^I - m s^I_= + \nabla_= \xi^I - m \nabla_J s^I_= \xi^J + O(\xi^2) \ , \tag{4.9}$$

hence

$$B^I_{=\ J} = \frac{\delta\pi^I_=}{\delta\xi^J} = \nabla_= \delta^I{}_J - m\nabla_J s^I_= \ . \tag{4.10}$$

Performing the $H^I_{\neq}$ integration, we obtain a delta function which projects the $\pi^I_=$ integration down on to the space of instanton solutions

$$\partial_= \phi^I - m s^I_= = 0 \ . \tag{4.11}$$

Furthermore, if we now use the freedom to take the limit $m \to \infty$, the ϕ^I fields become localized at the zeros of $s^I_=$. If we assume $s^I_=$ has discrete zeros, then the integral over $\pi^I_=$ becomes a sum over the zeros of $s^I_=$. We therefore have

$$\begin{aligned} &< \mathcal{O}^{(0)}_A > \\ &= \sum_{zeros} \int_M d\phi_0 \int d[\psi] d[\eta] \frac{\mathcal{O}^{(0)}_A}{\mid \det'(B^I_{=\ J}) \mid} exp(i \int d^2x g_{\bar{I}J} \psi^{\bar{I}}_{\neq} (\nabla_= \delta^J{}_K - m\nabla_K s^J_=)\eta^K) \ . \end{aligned} \tag{4.12}$$

Before continuing with the calculation we should make some remarks about zero modes and the ghost number anomaly. As we have just seen there are potentially bosonic zero modes ϕ^I_0 of $B^I_{=\ J}$. In addition there may also be fermionic zero modes η^I_0 and $\psi^I_{\neq 0}$. Indeed, the integrand in (4.12) is just $g_{\bar{I}J}\psi^{\bar{I}}_{\neq} B^J_{=\ K}\eta^K$. Now $B^J_{=\ K}$ and its adjoint define maps

$$\begin{aligned} B &: \Phi^* T\mathcal{M} \to \Lambda^{(1,0)}(\Sigma) \otimes \Phi^* T\mathcal{M} \ , \\ B^\dagger &: \Lambda^{(0,1)}(\Sigma) \otimes \Phi^* T\mathcal{M} \to \Phi^* T\mathcal{M} \ , \end{aligned} \tag{4.13}$$

which will generally have η^I_0 and $\psi^{\bar{I}}_{\neq 0}$ zero modes respectively. The number of these modes will in general depend upon the 1-form s and the topology of Σ. In order not to commit ourselves to a particular model, we will not discuss in any more detail here the existence of infinitesimal fermionic zero modes.

There is, however, a global obstruction to constructing finite zero modes from the infinitesimal ones above [6]. This causes the number of finite $B^I_{= \ J}$ zero modes to be given by the index

$$\mathrm{ind}B = \mathrm{dimKer}B - \mathrm{dimKer}B^\dagger \ , \tag{4.14}$$

which is interpreted as the virtual dimension of the moduli space M and is not, in general, a constant over M. The effect of the ghost number anomaly (4.14) is to give non vanishing expectation values to observables with non zero ghost number. As the observable (4.1) has ghost number n, we need n η^I_0 modes in the path integral to obtain a non vanishing vacuum expectation value. In addition, for these observables the presence of any $\psi^{\bar{I}}_{\neq 0}$ modes would cause the integral (4.6) to vanish. Hence we will assume that there are only ϕ^I and η^I zero modes.

Continuing with the calculation and integrating over the anticommuting non zero modes in (4.12) we obtain, by a standard result of Grassmann integration,

$$\begin{aligned} < \mathcal{O}^{(0)}_A > &= \sum_{zeros} \int_{\hat{M}} d\eta_0 \ d\phi_0 \frac{\det'(B^I_{= \ J})}{|\det'(B^I_{= \ J})|} \mathcal{O}^{(0)}_A \\ &= \sum_{zeros} \int_{\hat{M}} d\eta_0 \ d\phi_0 \ \mathrm{sgndet}'(B^I_{= \ J}) A_{I_1 \ldots I_n}(\phi_0) \eta^{I_1}_0 \ldots \eta^{I_n}_0 \ , \end{aligned} \tag{4.15}$$

where $\hat{M}$ is the supermoduli space which includes the anticommuting fields and can be viewed as the tangent bundle to M since we are assuming there are no $\psi^{\bar{I}}_{\neq}$ zero modes. As $m \to \infty$, $B^I_{= \ J}$ is dominated by the mass term. If we deform $g_{I\bar{J}}$ so that it is flat near the zeros of $s^I_=$ we obtain

$$\begin{aligned} < \mathcal{O}^{(0)}_A > &= \sum_{zeros} \mathrm{sgndet}'(\partial_J s^I_=) \int_{\hat{M}} d\eta_0 \ d\phi_0 A_{I_1 \ldots I_n}(\phi_0) \eta^{I_1}_0 \ldots \eta^{I_n}_0 \\ &= \sum_{zeros} \mathrm{sgndet}'(\partial_J s^I_=) \int_{M_n} d\phi_0 \ A_{I_1 \ldots I_n}(\phi_0) \ , \end{aligned} \tag{4.16}$$

where M_n is the n dimensional component of the moduli space M and we have used the canonical measure $d\eta_0 = d\eta^{I_1}_0 \ldots d\eta^{I_n}_0$ for the Grassmann integral. What

are we to make of the expression (4.16)? In the limit that $m \to \infty$, the zero modes of $B^I{}_J$ are generated by the directions in $\mathcal{M}$, at a given zero, along which $s^I_=$ is flat. We therefore interpret the integral in (4.16) to be over the n dimensional submanifold of $\mathcal{M}$ generated by the n flat directions of $s^I_=$ at each zero.

From (4.16) one can read off the partition function Z by considering the n=0 case, with $A(\phi) \equiv 1$ and no zero modes. Then in (4.16), the left hand side is just Z, while the right hand side simply becomes a weighted sum over the zeros of $s^I_=$

$$\begin{aligned} Z &= \sum_{zeros} \mathrm{sgndet}(\partial_J s^I_=) \\ &= \chi(\mathcal{M}) \ , \end{aligned} \tag{4.17}$$

which, by the Hopf Index Theorem, is just the Euler number of $\mathcal{M}$.

Having performed the above calculation it is effortless to consider the (2,2) case where $\psi^I_=, H^I_=, \eta^{\bar{I}} \neq 0$. Here we simply repeat the calculation with $(\psi^{\bar{I}}_{\neq}, H^{\bar{I}}_{\neq}, \eta^I) \leftrightarrow (\psi^I_=, H^I_=, \eta^{\bar{I}})$ and combine the two results. Equation (4.16) becomes

$$< \mathcal{O}^{(0)}_A >= \sum_{zeros} \mathrm{sgndet}'(\partial_j s^i) \int_{M_n} d\phi_0 \ A_{i_1 \ldots i_n}(\phi_0) \ , \tag{4.18}$$

where $\mathrm{sgndet}'(\partial_j s^i) = \mathrm{sgndet}'(\partial_{\bar{J}} s^{\bar{I}}_{\neq})\mathrm{sgndet}'(\partial_J s^I_=)$ and we sum over the zeros of s^i (i.e. the common zeros of $s^I_=$ and $s^{\bar{I}}_{\neq}$).

Thus we have reduced the calculation of $< \mathcal{O}^{(0)}_A >$ to (4.16) and recovered the standard result that the partition function is equal to the Euler number of $\mathcal{M}$. In the case where $s^I = \partial^I f$ for some f, so that the sigma model (3.4) possesses (2,2) supersymmetry, our assumptions about discrete zeros and no zero modes are then just that f is a Morse function on $\mathcal{M}$ (i.e. it has discrete, nondegenerate extrema).

The result (4.17) is then related to the Morse formula for the Euler number, *viz*

$$
\begin{aligned}
Z = \chi(\mathcal{M}) &= \sum_{extrema} \mathrm{sgndet}(\partial_J \partial^I f) \\
&= \sum_{det>0} 1 \; - \; \sum_{det<0} 1 \\
&= \sum_{n=0}^{D/2} M_{2n} - \sum_{n=0}^{D/2} M_{2n+1} \\
&= \sum_{n=0}^{D} (-1)^n M_n \ ,
\end{aligned}
\tag{4.19}
$$

where M_n is the nth Morse number (i.e. the number of extrema of f with n negative modes) and D is the (real) dimension of $\mathcal{M}$.

Finally, it is instructive to rewrite the action (3.9), when $\mathcal{M}$ is complex and $g_{I\bar{J}}$ Hermitian, as

$$
\begin{aligned}
S_{top} = \int d^2x \Big\{ & \frac{1}{\alpha} g_{I\bar{J}} (\partial_= \phi^I - m s^I_=)(\partial_{\neq} \phi^{\bar{J}} - m s^{\bar{J}}_{\neq}) \\
& - i g_{\bar{I}J} \psi^{\bar{I}}_{\neq} (\nabla_= \eta^J - m \nabla_K s^J_= \eta^K) - i g_{I\bar{J}} \psi^I_= (\nabla_{\neq} \eta^{\bar{J}} - m \nabla_{\bar{K}} s^{\bar{J}}_{\neq} \eta^{\bar{K}}) \\
& + \alpha \psi^I_= \psi^{\bar{I}}_{\neq} R_{I\bar{I}J\bar{J}} \eta^J \eta^{\bar{J}} \Big\} \ .
\end{aligned}
\tag{4.20}
$$

From this it becomes clear that the path integral is dominated, for small α, by the instanton solutions (4.11). If we ignore the connection terms in (4.20) and identify $(\phi^I, \phi^{\bar{I}}, \eta^I, \eta^{\bar{I}}, \psi^I_=, \psi^{\bar{I}}_{\neq})$ with the fields $(U^I, U^{\bar{I}}, \chi^I, \chi^{\bar{I}} \rho^I_z, \rho^{\bar{I}}_{\bar{z}})$ of reference [5] we arrive at the (2,2) supersymmetric topological Landau-Ginzburg model of reference [5], with the potential W given by f and suitably interpreted as a world sheet 1-form. In the case where we set $\psi^I_= = \eta^{\bar{I}} = 0$, the model is a twisted form of the (2,0) Landau-Ginzburg model. Thus the topological Landau-Ginzburg theories arise simply from the topological massive sigma model in the limit where the target space can be considered flat. In fact if the $\mathcal{M}$ admits a flat metric (i.e. if it has the topology of $\mathbf{C}^{D/2}$, possibly with points removed) then we may make the "gauge"

choice $g_{ij} = \delta_{ij}$ in which case the massive topological sigma model is simply the topological Landau-Ginzburg theory.

In [5,9] it was shown that the observables of the topological Landau-Ginzburg model and topological massless sigma model both have the same representation in bosonic fields as the primary fields of N=2 superconformal field theories. Above we calculated the expectation values of some observables of the topological sigma model using the freedom to let $m \to \infty$. This had the effect of allowing us to ignore the metric structure of the target manifold (although the fields were still constrained to lie in the target space). However, when we ignore the target space metric, the model becomes the topological Landau-Ginzburg theory. Thus the topological massive sigma model constructed here can be viewed as interpolating between the massless topological sigma model of [2] at $m = 0$ and the topological Landau-Ginzburg model of [5] as $m \to \infty$. In the infinite mass limit the ϕ^i are localized about the zeros of s^i which are the vacuum states of the corresponding Landau-Ginzburg model. Furthermore, in order that the observables $\mathcal{O}_A$ have a non zero expectation value, s^i must possess "flat" directions. In the Landau-Ginzburg model these flat directions produce the massless excitations which are needed for the associated conformal field theory to be nontrivial.

5. **Comments**

In this paper we constructed a topological massive sigma model related to the twisted massive (2,0) supersymmetric sigma model. In addition to providing non trivial potentials for the bosonic fields, the inclusion of mass terms in the topological sigma model allows one to simplify the calculation of some observables. Furthermore, we argued that in the limit $m \to \infty$ the topological massive sigma model becomes a topological Landau-Ginzburg model. Hence the topological sigma model and the topological Landau-Ginzburg model may be viewed as the same theory, interpolated by the topological massive sigma model. We have only presented one possible way of constructing topological massive sigma models here. It would

be of interest to pursue other massive models, in particular (2,0) models where the vector bundle is not associated with the tangent space. In addition it would be interesting to couple the model here to 2-D topological gravity and investigate the resulting string theory and "space of all 2D topological field theories".

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